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Optimum Spatial Filter and Uncertainty

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This paper deals with a basic aspect of spatial filters that analyze spatial waveforms such as visual patterns. In this paper, the n-dimensional spatial filters are assumed to be linear, homogeneous, isotropic and low-pass. The uncertainty relations between spatial resolution and spatial frequency resolution of the spatial filters are obtained by means of the variational method. The optimum filters in the sense of minimizing the product of their resolutions are obtained. The following three cases are considered: *Case 1.* In this case, the impulse response $f(\mathbf{x})$ and the transfer function $\phi(\boldsymbol{\xi})$ are allowed to spread over the entire spatial domain E^n and spatial angular frequency domain $\mathcal{E}^n$, respectively. *Case 2.* In this case, $f(\mathbf{x})$ vanishes outside the hypersphere of radius R with the center at the origin of E^n. *Case 3.* In this case, $\phi(\boldsymbol{\xi})$ vanishes outside the hypersphere of radius P with the center at the origin of $\mathcal{E}^n$.

I. INTRODUCTION

The uncertainty principle in quantum mechanics was advocated by Heisenberg (1927) and formulated mathematically by Weyl (Weyl, 1928; Margenau and Murphy, 1943). Since Gabor (1946) had applied the Weyl formulation of the uncertainty principle to communication theory, optimum pulse-shape problems were considered by Landau and Pollak (1961), Hosono and Oowaku (1965) and many others from the point of view of signal design.

On the other hand, filters with high resolution in both the time and frequency domains are necessary for the signal analysis. Problems of the optimum filter design in the sense of minimizing the product of their resolutions are similar to the above problems in their mathematical form. Consequently the optimum filter obtained by Gabor (1946) has been used in the field of speech-sound analysis.

However, the present input to this Gaussian filter affects not only the future output but also the past output. That is, the law of causality is not satisfied for this filter. One author has pointed this out and obtained the physically realizable optimum filter for which the law of causality is satisfied (Isomichi, 1966).

Mean frequency for a filter, *i.e.* the center of the frequency-pass-band of the filter, had been defined as the first moment of the square of the transfer function. Since the impulse response of any filter is a real valued function, the mean frequency defined above becomes zero. Hence, only low-pass filters had been able to be dealt with heretofore. One of the authors, however, introduced a new signal-analysis method which is called "analytic spectrum method". He also obtained the physically realizable optimum band-pass filter using the method (Isomichi, 1967; Isomichi and Iijima, 1968). The studies of temporal filters have been completed in this way.

Recently, new information processing systems have been required in the research of pattern recognition. And spatial filters become necessary for the analysis of spatial waveforms analysis. Hence, it is required to find uncertainty relations between spatial resolution and spatial frequency resolution and to find optimum characteristics of spatial filters in the sense of minimizing the product of their resolutions.[1]

There are two large differences between the spatial filter and the temporal filter. First, the former is not restricted by the law of causality which is required for the latter. Second, the spatial filter presents the problem of the dimension of the space on which signals are accepted by the spatial filter as its input. Hence, the "analytic spectrum method" cannot be applied to n-dimensional spatial filters straightforwardly.

Therefore, as the first step toward the consideration of band-pass spatial filters, this paper deals with the n-dimensional spatial filters, which are linear, homogeneous, isotropic and low-pass. They accept spatial waveforms on the n-dimensional Euclidean space E^n as their input. The problems are formulated as the variational problems of

[1] In the middle of the 1950's many energetic researchers applied communication theory to optics (Kubota, 1963). And it has been already obtained at that time that the uncertainty relation between coherence time $\Delta\tau$ of polychromatic radiation and variance $\Delta\nu$ of the average spectral energy density is of the form $\Delta\tau \cdot\Delta\nu \geqq (4\pi)^{-1}$ (Wolf, 1958). However, the inequality is too "weak" and the equality sign cannot be achieved.

This inequality gives the uncertainty relation not of the optical system but of the optical image. It is important to obtain the uncertainty relations of the optical systems. However, none has examined this problem up-to-date.

determining the impulse response $f(\mathbf{x})$ of the spatial filter such that, for a given spread of $f(\mathbf{x})$, the spread of its transfer function $\phi(\boldsymbol{\xi})$ is minimum, where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ are n-dimensional vectors in the spatial domain E^n and the spatial angular frequency domain $\mathcal{E}^n$, respectively. Since the spatial filter is homogeneous, its impulse response $f(\mathbf{x}, \mathbf{y})$ becomes the function $f(\mathbf{x} - \mathbf{y})$ which depends only on the vector $\mathbf{x} - \mathbf{y}$. Moreover, since the filter is isotropic, $f(\mathbf{x})$ depends only on the distance of $\mathbf{x}$ from the origin, namely, $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$; we call such functions $f(\mathbf{x})$ radial functions. The following three cases are considered:

1. *The case of "infinite area."* In this case, $f(\mathbf{x})$ and $\phi(\boldsymbol{\xi})$ are allowed to spread over entire E^n and $\mathcal{E}^n$, respectively.
2. *The case of "space limitation."* In this case, $f(\mathbf{x})$ vanishes outside the hypersphere of radius R with the center at the origin of E^n, which is denoted by $S_n(R)$.
3. *The case of "frequency limitation."* In this case, $\phi(\boldsymbol{\xi})$ vanishes outside the hypersphere of radius P with the center of the origin of $\mathcal{E}^n$, which is denoted by $\Sigma_n(P)$.

The third case is similar to the second case in mathematical forms. Hence, as to the third case, only the conclusions are given.

II. ASSUMPTION AND MATHEMATICAL PRELIMINARY

Let the impulse response $f(\mathbf{x})$ of a spatial filter satisfy the following two conditions:

(i) The impulse response $f(\mathbf{x})$ is a real valued function of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in E^n.

(ii) The impulse response $f(\mathbf{x})$ is a radial function, which is denoted by $f(r)$ in the polar coordinates, where

$$r = \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The assumption (i) implies that $f(\mathbf{x})$ is actually observable. The assumption (ii) implies that the filter in question is isotropic. The function $f(\mathbf{x})$ connects the input and the output of the filter in the form established in the footnote 3. Hence, whenever the impulse response is denoted by $f(\mathbf{x})$ it is already assumed that the filter is linear and homogeneous.

The following five assumptions are also adopted.

(iii) $\int_{E^n} \{f(\mathbf{x})\}^2 \, d\mathbf{x} < \infty.$

(iv) $\int_{E^n} \left\{ \frac{\partial}{\partial x_j} f(\mathbf{x}) \right\}^2 d\mathbf{x} < \infty, \qquad (j = 1, 2, \cdots, n).$

(v) $\int_{E^n} \left\{ \frac{\partial^2}{\partial x_j \partial x_k} f(\mathbf{x}) \right\}^2 d\mathbf{x} < \infty, \qquad (j, k = 1, 2, \cdots, n).$

(vi) $f(\mathbf{x})$: absolutely continuous in every closed and bounded domain (Smirnov, 1964, paragraph 77).

(vii) $\partial F(\mathbf{x})/\partial x_j$ $(j = 1, 2, \cdots, n)$: absolutely continuous in every closed and bounded domain (or in $S_n(R)$ in the case of "space limitation")[2].

From the assumption (iii), $f(\mathbf{x})$ is Fourier transformable and Parseval's formula holds (Bochner and Chandrasekharan, 1949, p. 120):

$$\phi(\xi) = \underset{a\to\infty}{\text{l.i.m.}} \frac{1}{V_n(s)} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\|\mathbf{x}\| \leq a} f(\mathbf{x}) e^{-i(\xi\cdot\mathbf{x})} \, d\mathbf{x}, \tag{1}$$

$$f(\mathbf{x}) = \underset{\alpha\to\infty}{\text{l.i.m.}} V_n(s) \left(\frac{1}{2\pi}\right)^{n/2} \int_{\|\xi\| \leq \alpha} \phi(\xi) e^{i(\xi\cdot\mathbf{x})} \, d\xi, \tag{2}$$

$$\frac{1}{V_n(s)} \int_{E^n} \{f(\mathbf{x})\}^2 \, d\mathbf{x} = V_n(s) \int_{\mathcal{E}^n} \{\phi(\xi)\}^2 \, d\xi, \tag{3}$$

where $(\xi\cdot\mathbf{x}) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $\|\mathbf{x}\| = \sqrt{(\mathbf{x}\cdot\mathbf{x})}$. The notation l.i.m. implies mean convergence (Smirnov, 1964, paragraph 56). The notation $V_n(s)$ implies the volume of a n-dimensional hypersphere of radius s and is expressed, using n-dimensional hyper-solid-angle Ω_n, as follows:

$$V_n(s) = \frac{\Omega_n s^n}{n}, \qquad \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{4}$$

where $\Gamma(z)$ is the Gamma function. And s is the square root of the centered second moment of $\{f(\mathbf{x})\}^2$ (See Eq. (12)).

In this paper, Fourier transforms defined by Eq. (1) differ from the ordinary definition in the term of $1/V_n(s)$. This definition may be regarded as ordinary Fourier transform normalized per unit hypersphere volume. However, it is important that the dimension $[L]^n$ of $d\mathbf{x}$ offsets

[2] The family of functions restricted by assumptions (iii)–(vii) is denoted by $W_2^{(2)}(E^n)$ after Sobolev (Smirnov, 1964, paragraph 112 and 110). The assumption (iv) is deduced from assumptions (iii), (v) and (vi) (Bochner and Chandrasekharan, 1949, p. 129). The assumption (v) becomes unnecessary in the case of "space limitation."

the dimension $[L]^{-n}$ of $1/V_n(s)$ and the integral operator (1) becomes the dimensionless operator which transforms a dimensionless function $f(\mathbf{x})$ to a dimensionless function $\phi(\xi)$. This fact holds for the Fourier inverse transform (2). Hereinafter such dimensionless integrals are used.[3]

From assumptions (iii), (iv) and (vi) it follows that

$$i\xi\phi(\xi) = \underset{a\to\infty}{\text{l.i.m.}} \frac{1}{V_n(s)} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\|\mathbf{x}\| \leq a} \nabla f(\mathbf{x}) e^{-i(\xi\cdot\mathbf{x})}\, d\mathbf{x},$$

where ∇ is so-called "nabla" of the n-dimensional vector operator, that is,

$$\nabla f(\mathbf{x}) = e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} + \cdots + e_n \frac{\partial f}{\partial x_n},$$

where e_j are unit vectors forming a basis for E^n. Hence, by Parseval's formula it follows that

$$V_n(s) \int_{\mathcal{E}^n} \|\xi\|^2 \cdot |\phi(\xi)|^2\, d\xi = \frac{1}{V_n(s)} \int_{E^n} \|\nabla f(\mathbf{x})\|^2\, d\mathbf{x}. \tag{5}$$

From assumptions (i)–(iii), $\phi(\xi)$ is a real valued radial function, which is denoted by $\phi(\rho)$ where $\rho = \|\xi\|$, and we then have the Hankel transform of order ν

$$\phi(\rho) = \underset{R\to\infty}{\text{l.i.m.}} \frac{1}{V_n(s)} \frac{1}{\rho^\nu} \int_0^R r^{\nu+1} f(r) J_\nu(\rho r)\, dr \tag{6}$$

corresponding to the Fourier transform (1), where $J_\nu(z)$ is the Bessel function of order $\nu = (n-2)/2$ (Bochner and Chandrasekharan, 1949, p. 122). And it follows that

$$\begin{cases} \dfrac{1}{V_n(s)} \displaystyle\int_{E_n} \mathbf{x}\{f(\mathbf{x})\}^2\, d\mathbf{x} = 0 \\ V_n(s) \displaystyle\int_{\mathcal{E}^n} \xi\{\phi(\xi)\}^2\, d\xi = 0 \end{cases} \tag{7}$$

[3] Therefore, the output $g_o(\mathbf{x})$ of the spatial filter is related with the input $g_i(\mathbf{x})$ through its impulse response $f(\mathbf{x})$ as follows:

$$g_o(\mathbf{x}) = \frac{1}{V_n(s)} \int_{E^n} f(\mathbf{x} - \mathbf{y}) g_i(\mathbf{y})\, d\mathbf{y}.$$

We are indebted to Dr. Iijima for these modified expressions of Fourier transforms.

From assumptions (ii), (vi) and (vii), $f(r)$ and $f'(r) = df(r)/dr$ are absolutely continuous in every bounded interval, and hence $r^{n-1}f'(r)$ is absolutely continuous in every bounded interval (Smirnov, 1964, paragraph 74).

From assumptions (ii), (iii), (iv) and (vi), we have

$$r^{(n-1)/2}f(r) \to 0 \, (r \to \infty) \tag{8}$$

(Smirnov, 1964, paragraph 188). From assumptions (ii), (iv), (v) and (vii), we have

$$r^{(n-1)/2}f'(r) \to 0 \;\; (r \to \infty). \tag{9}$$

Using these preliminaries, uncertainty relations and optimum characteristics of spatial filters are considered in the following sections.

III. THE CASE OF INFINITE AREA

By Eq. (7), the first moments of both $\{f(\mathbf{x})\}^2$ and $\{\phi(\boldsymbol{\xi})\}^2$ turn out to be zero. Hence the spreads of $f(\mathbf{x})$ and $\phi(\boldsymbol{\xi})$ are defined as the square root of the centered second moment of $\{f(\mathbf{x})\}^2$ and $\{\phi(\boldsymbol{\xi})\}^2$, which are denoted by s and σ, respectively (See Eqs. (12) and (10)). Then, by the assumption (ii) being considered, the problem is formulated as that of minimizing the following equation (10) under subsidiary conditions (11) and (12):

$$\sigma^2 = J[f] = \frac{\int_{\mathcal{E}^n} \|\boldsymbol{\xi}\|^2 \{\phi(\boldsymbol{\xi})\}^2 \, d\boldsymbol{\xi}}{\int_{\mathcal{E}^n} \{\phi(\boldsymbol{\xi})\}^2 \, d\boldsymbol{\xi}} \tag{10}$$

conditions:

$$\begin{cases} \dfrac{1}{V_n(s)} \displaystyle\int_{E^n} \{f(\mathbf{x})\}^2 \, d\mathbf{x} = A^2 & (11) \\[2ex] \dfrac{\displaystyle\int_{E^n} \|\mathbf{x}\|^2 \{f(\mathbf{x})\}^2 \, d\mathbf{x}}{\displaystyle\int_{E^n} \{f(\mathbf{x})\}^2 \, d\mathbf{x}} = s^2 & (12) \end{cases}$$

Rewriting Eq. (10) in terms of $f(\mathbf{x})$ using Eqs. (3) and (5), we have

$$\sigma^2 = J\,[f] = \frac{\dfrac{1}{V_n(s)} \displaystyle\int_{E^n} \|\nabla f(\mathbf{x})\|^2 \, d\mathbf{x}}{\dfrac{1}{V_n(s)} \displaystyle\int_{E^n} \{f(\mathbf{x})\}^2 \, d\mathbf{x}}, \tag{13}$$

so that, from the assumption (ii), the problem is rewritten in terms of $f(r)$ as follows.

$$\sigma^2 = J[f] = \frac{n}{A^2 s^n} \int_0^\infty r^{n-1}\{f'(r)\}^2\, dr \tag{14}$$

conditions:

$$\begin{cases} \dfrac{n}{A^2 s^n} \displaystyle\int_0^\infty r^{n-1}\{f(r)\}^2\, dr = 1 & (15) \\ \dfrac{n}{A^2 s^n} \displaystyle\int_0^\infty r^{n+1}\{f(r)\}^2\, dr = s^2 & (16) \\ f'(0) = 0 & (17) \\ r^{(n-1)/2} f(r) \to 0 \ (r \to \infty) & (18) \end{cases}$$

The boundary condition (17) is deduced from the continuity of $\partial f(\mathbf{x})/\partial x_j$ $(j = 1, 2, \cdots, n)$ at $x = 0$, which is deduced from assumptions (ii) and (vii). The boundary condition (18) is the same as Eq. (8).

In order to solve the variational problem, let us calculate the variation of $J[f]$ under subsidiary conditions (15) and (16). Then we have

$$\begin{aligned} \delta J = \underset{R\to\infty}{\mathrm{l.i.m.}} \frac{2n}{A^2 s^n} \Bigg[&\int_0^R r^{n-1} f'(r) \delta f'(r)\, dr \\ &+ \lambda_1 \frac{2n}{A^2 s^n} \int_0^R r^{n-1} f(r) \delta f(r)\, dr \\ &+ \lambda_2 \frac{2n}{A^2 s^n} \int_0^R r^{n+1} f(r) \delta f(r)\, dr \Bigg], \end{aligned} \tag{19}$$

where λ_1 and λ_2 are Lagrange's multipliers, and $\delta f(r)$ and $\delta f'(r)$ are variations of $f(r)$ and $f'(r)$, respectively. Since $r^{n-1} f'(r)$ and $\delta f(r)$ are absolutely continuous in the interval $[0, R]$, we can integrate by parts the first term on the right-hand side of Eq. (19), and hence

$$\begin{aligned} \delta J = \underset{R\to\infty}{\mathrm{l.i.m.}} \frac{2n}{A^2 s^n} \Bigg[& r^{n-1} f'(r) \delta f(r) \Big|_0^R \\ & - \int_0^R \{ r^{n-1} f''(r) + (n-1) r^{n-2} f'(r) \} \delta f(r)\, dr \\ & + \int_0^R (\lambda_1 r^{n-1} + \lambda_2 r^{n+1}) f(r) \delta f(r)\, dr \Bigg]. \end{aligned} \tag{20}$$

Now $\delta f(r)$ is bounded in $[0, R]$ because of the absolute continuity of $\delta f(r)$ in $[0, R]$ so that by Eq. (17)

$$r^{n-1} f'(r) \delta f(r) \big|_{r=0} = 0. \tag{21}$$

By Eq. (8) and (9), it follows that

$$r^{n-1}f'(r)\delta f(r) \to 0 \qquad (r \to \infty). \tag{22}$$

From Eqs. (21) and (22), the first term on the right-hand side of Eq. (20) becomes zero, so that

$$\begin{aligned}\delta J = (-)\frac{2n}{A^2 s^n}\int_0^\infty \{rf''(r) + (n-1)f'(r) \\ - (\lambda_1 r + \gamma_2 r^3)f(r)\}r^{n-2}\delta f(r)\, dr = 0.\end{aligned} \tag{23}$$

Since Eq. (23) must be valid for any arbitrary function $\delta f(r)$ which satisfies assumptions (i)–(vii) in Section II, we have then the following Euler's differential equation

$$rf''(r) + (n-1)f'(r) - (\lambda_1 r + \lambda_2 r^3)f(r) = 0. \tag{24}$$

Now the problem discussed above has been reduced to solving the differential equation (24) under boundary conditions (17) and (18). If we write

$$f(r) \sim r^m \exp(-pr^q)$$

and examine the behavior of $f(r)$ as r tends to infinity, it is found that, since $f(r) \to 0$ as $r \to \infty$,

$$\begin{cases} m\text{: arbitrary nonnegative integer,} \\ q = 2, \\ p = \sqrt{\lambda_2}/2 > 0. \end{cases} \tag{25}$$

Hence, if we write

$$f(r) = u(r)\exp\left(-\frac{\sqrt{\lambda_2}}{2}r^2\right), \tag{26}$$

then Eq. (24) becomes

$$ru''(r) + (n - 1 - 2\sqrt{\lambda_2}r^2)u'(r) - (\lambda_1 + n\sqrt{\lambda_2})ru(r) = 0, \tag{27}$$

so that, if in this equation we make the substitutions

$$\begin{cases} x = \sqrt{\lambda_2}\, r^2 \geqq 0, \\ \lambda = (-)\dfrac{\lambda_1 + n\sqrt{\lambda_2}}{4\sqrt{\lambda_2}}, \quad \nu = \dfrac{n-2}{2}, \\ v(\sqrt{\lambda_2}\, r^2) = u(r), \end{cases} \tag{28}$$

then we find that Eq. (27) becomes

$$xv''(x) + (\nu + 1 - x)\,v'(x) + \lambda v(x) = 0. \tag{29}$$

By Eqs. (26) and (28) it follows that

$$\begin{cases} f(r) = v(x) \exp(-x/2), \\ f'(r) = \sqrt{2px}\,\{2v'(x) - v(x)\} \exp(-x/2), \end{cases}$$

and hence the boundary conditions corresponding to boundary conditions (17) and (18) become

$$\begin{cases} |\,2v'(0) - v(0)\,| < \infty, \\ v(x) = O(x^N)\,(x \to \infty), \end{cases} \tag{30}$$

where N is a nonnegative integer. Under boundary conditions (30), the associated Laguerre's differential equation (29) has solutions

$$L_N^{(\nu)}(x) = \sum_{k=0}^{N} (-1)^k \binom{N+\nu}{N-k} \frac{x^k}{k!}, \qquad N = 0, 1, 2, \cdots \tag{31}$$

for and only for $\lambda = N$, where $L_N^{(\nu)}(x)$ are associated Laguerre's polynomials. Hence, if C is an arbitrary real constant, solutions of Eq. (24) are

$$f(r) = \mathrm{C}\, L_N^{(\nu)}(\sqrt{\lambda_2}\, r^2) \exp\left(-\frac{\sqrt{\lambda_2}}{2} r^2\right), \qquad N = 0, 1, 2, \cdots. \tag{32}$$

Substituting Eq. (32) into Eq. (15) in order to obtain undetermined constants C and λ_2, we have

$$\begin{aligned} \frac{A^2 s^n}{n} &= \frac{\mathrm{C}^2}{2(\sqrt{\lambda_2})^{\nu+1}} \int_0^\infty r^\nu e^{-r} \{L_N^{(\nu)}(r)\}^2\, dr \\ &= \frac{\mathrm{C}^2}{2(\sqrt{\lambda_2})^{\nu+1}} \frac{(N+\nu+1)}{N!} \end{aligned} \tag{33}$$

(Sansone, 1959, p. 302), so that, by Eq. (4) it follows that

$$\mathrm{C} = \pm A \sqrt{\frac{\Omega_n}{n} \frac{N!\Gamma(\nu+1)}{\Gamma(N+\nu+1)} \left(\frac{s^2\sqrt{\lambda_2}}{\pi}\right)^{\nu+1}}. \tag{34}$$

By Eqs. (32) and (16), it follows that

$$\begin{aligned} \frac{A^2 s^{n+2}}{n} &= \frac{\mathrm{C}^2}{2(\sqrt{\lambda_2})^{\nu+2}} \int_0^\infty r^{\nu+1} e^{-r} \{L_N^{(\nu)}(r)\}^2\, dr \\ &= \frac{\mathrm{C}^2}{2(\sqrt{\lambda_2})^{\nu+2}} \int_0^\infty r^\nu e^{-r} L_N^{(\nu)}(r)[(2N+\nu+1)L_N^{(\nu)}(r) \\ &\quad - (N+1)L_{N+1}^{(\nu)}(r) - (N+\nu)L_{N-1}^{(\nu)}(r)]\, dr \end{aligned}$$

(Sansone, 1959, p. 297)

$$= \frac{C^2(2N + \nu + 1)\Gamma(N + \nu + 1)}{2(\sqrt{\lambda_2})^{\nu+2} N!}$$

(Sansone, 1959, p. 301), and by Eq. (33)

$$= \frac{2N + \nu + 1}{\sqrt{\lambda_2}} \frac{A^2 s^n}{n}.$$

Hence, it follows that

$$\sqrt{\lambda_2} = \frac{4N + n}{2s^2}. \tag{35}$$

By Eqs. (34) and (35),

$$C = \pm A \sqrt{\frac{\Omega_n}{n} \frac{N!\Gamma(\nu + 1)}{\Gamma(N + \nu + 1)} \left(\frac{4N + n}{2\pi}\right)^{n/2}}. \tag{36}$$

It is not essential in these problems whether C is positive or negative, and hence the positive sign is accepted in this paper. Substituting Eqs. (35) and (36) into Eq. (32), we have

$$\begin{aligned} f_N(r) = A \sqrt{\frac{\Omega_n}{n} \frac{N!\Gamma(\nu + 1)}{\Gamma(N + \nu + 1)}} \left(\frac{4N + n}{2\pi}\right)^{n/4} \\ \cdot L_N^{(\nu)}\left(\frac{4N + n}{2}\left(\frac{r}{s}\right)^2\right) \exp\left(-\frac{1}{2}\frac{4N + n}{2}\left(\frac{r}{s}\right)^2\right), \\ \nu = (n - 2)/2, \quad N = 0, 1, 2, \cdots. \end{aligned} \tag{37}$$

These functions satisfy all assumptions in Section II certainly.

Multiplying Eq. (24) by $(n/A^2s^n)r^{n-2}f(r)$, integrating from 0 to R and noting Eqs. (14), (15), and (16), we have

$$\begin{aligned} \lambda_1 + \lambda_2 s^2 \\ = \underset{R\to\infty}{\text{l.i.m.}} \frac{n}{A^2 s^n} \int_0^R [r^{n-1}f''(r)f(r) + (n - 1)r^{n-2}f'(r)f(r)]\, dr \\ = \underset{R\to\infty}{\text{l.i.m.}} \frac{n}{A^2 s^n} \int_0^R \left[\frac{d}{dr}\{r^{n-1}f'(r)f(r)\} - r^{n-1}\{f'(r)\}^2\right] dr \\ = \underset{R\to\infty}{\text{l.i.m.}} \frac{n}{A^2 s^n} \int_0^R \left[\frac{d}{dr}\{r^{n-1}f'(r)f(r)\}\right] dr - J[f]. \end{aligned} \tag{38}$$

The function $r^{n-1}f'(r)f(r)$ is absolutely continuous in $[0, R]$ because of absolute continuity of $r^{n-1}f'(r)$ and $f(r)$ in $[0, R]$ (Smirnov, 1964, paragraph 74). Hence, it follows that

$$J[f] = \underset{R\to\infty}{\text{l.i.m.}} \frac{n}{A^2 s^n} [r^{n-1}f'(r)f(r)]_0^R - \lambda_1 - \lambda_2 s^2. \tag{39}$$

The first term on the right-hand side of Eq. (39) becomes zero by the reasoning used at Eqs. (21) and (22), so that

$$J[f] = -\lambda_1 - \lambda_2 s^2. \tag{40}$$

By Eqs. (35) and (28), it follows for $\lambda = N$ that

$$\sigma_N^2 \equiv J[f_N] = \left(\frac{4N + n}{2s}\right)^2, \qquad N = 0, 1, 2, \cdots. \tag{41}$$

Since the value of Eq. (41) is minimum for $N = 0$, it follows for an arbitrary function $f(r)$ that

$$\sigma^2 = J[f] \geqq \left(\frac{n}{2s}\right)^2. \tag{42}$$

Hence the following uncertainty relation holds:

$$s\sigma \geqq \frac{n}{2}. \tag{43}$$

We shall call $\Delta_1 \equiv n/2$ "the uncertainty of the family of filters" in the case of infinite area.

Since the equality sign in Eq. (43) holds only for $N = 0$, putting $N = 0$ in Eq. (37) and noting that $L_0^{(\nu)}(x) = 1$ (Sansone, 1959, p. 296), we find that the optimum characteristic of the spatial filter in the case of infinite area is as follows:

$$f_0(r) = A \sqrt{\frac{\Omega_n}{n}} \left(\frac{n}{2\pi}\right)^{n/4} \exp\left(-\frac{1}{2}\frac{n}{2}\left(\frac{r}{s}\right)^2\right), r = \| \mathbf{x} \| \geqq 0. \tag{44}$$

Substituting Eq. (44) into Eq. (6) in order to obtain the optimum transfer function corresponding to $f_0(r)$, we find

$$\begin{aligned}\phi_0(\rho) &= \frac{A}{V_n^{(s)}\rho^\nu} \sqrt{\frac{\Omega_n}{n}} \left(\frac{n}{2\pi}\right)^{n/4} \\ &\quad \cdot \int_0^\infty r^{\nu+1} \exp\left(-\frac{n}{4s^2} r^2\right) J_\nu(\rho r)\, dr \\ &= A \sqrt{\frac{n}{\Omega_n}} \left(\frac{2}{n\pi}\right)^{n/4} \exp\left(-\frac{1}{2}\frac{2}{n}(s\rho)^2\right)\end{aligned}$$

by means of the formula (Watson, 1952, p. 394)

$$\int_0^\infty t^{\nu+1} \exp(-p^2t^2)J_\nu(at)\,dt = \frac{a^\nu}{(2p^2)^{\nu+1}} \exp\left(-\frac{a^2}{4p^2}\right), \quad (45)$$

which is valid for Re $(\nu) > -1$ (Re denoting "the real part of"). Hence, the optimum transfer function of the spatial filter in the case of infinite area is of the form

$$\phi_0(\rho) = A\sqrt{\frac{n}{\Omega_n}}\left(\frac{2}{n\pi}\right)^{n/4} \exp\left(-\frac{1}{2}\frac{2}{n}(s\rho)^2\right), \rho = \|\xi\| \geqq 0. \quad (46)$$

The calculation for general $\phi_N(\rho)$ is reserved for Appendix A.

The differences between the optimum spatial filters and the optimum temporal filter which has been already obtained by one of the authors (Isomichi, 1966)[4] are given in Tables I–III. In constructing these tables, we used the following formulae:

$$\Gamma\left(N+\frac{1}{2}\right) = \frac{(2N)!}{2^{2N}N!}\sqrt{\pi}, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (47)$$

(Abramowitz and Stegun, Ed., 1964, p. 255),

$$L_N^{(-1/2)}(x^2) = \frac{(-1)^N}{2^{2N}N!} H_{2N}(x), \quad (48)$$

$$L_N^{(1/2)}(x^2) = \frac{(-1)^N}{2^{2N+1}N!} \frac{H_{2N+1}(x)}{x}, \quad (49)$$

(Sansone, 1959, p. 319), where $H_N(x)$ are Hermite polynomials, that is,

$$H_N(x) = (-1)^N e^{x^2} \frac{d^N}{dx^N} e^{-x^2} \quad (50)$$

Variations of $f_0(r)$ and $\phi_0(\rho)$ are illustrated for $n = 1 \sim 3$ in Figs. 1 and 2, respectively.

Considering that filters are isotropic in this paper, let us pick up only even functions from the stationary functions of 1-dimensional spatial filters which have been obtained by Hosono and Oowaku (1965),[5] and

[4] In Tables I and III, A used in Isomichi's paper (1966) has been replaced by sA^2 according to the normalization method in this paper. Since the transfer functions are expressed not in terms of angular frequency but in terms of frequency in previous papers (Isomichi, 1966; Hosono and Oowaku, 1965), the uncertainties are $(2\pi)^{-1}$ times of those in this paper.

[5] Hosono and Oowaku (1965) used the terminology "pulse" in their paper. In this paper, however, the terminology "spatial filter" is used instead of the terminology "pulse."

let us renumber them. Then we find that they coincide with the stationary functions of 1-dimensional spatial filters in Table I except for the term $(-1)^N$. However, this difference is not essential.

The stationary functions of the variational problem for 3-dimensional

TABLE I

STATIONARY FUNCTIONS $f_N(r)$ AND $f_N(t)$ AND UNCERTAINTIES IN THE CASE OF "INFINITE AREA"

Dimension		Stationary functions $f_N(r)$, $f_N(t)$; $r = \|\mathbf{x}\| \geqq 0$, $t \geqq 0$, $N = 0, 1, 2, \cdots$ *, $\nu = \frac{n-2}{2}$	Uncertainties $s\sigma \geqq n/2$
Space	n	$A\sqrt{\frac{\Omega_n}{n}\frac{N!\Gamma(\nu+1)}{\Gamma(N+\nu+1)}}\left(\frac{4N+n}{2\pi}\right)^{n/4} \cdot L_N^{(\nu)}\left(\frac{4N+n}{2}\left(\frac{r}{s}\right)^2\right)\exp\left(-\frac{1}{2}\frac{4N+n}{2}\left(\frac{r}{s}\right)^2\right)$	$\Delta_1 = \frac{n}{2}$
	1	$A\frac{(-1)^N\sqrt{2}}{2^N\sqrt{(2N)!}}\left(\frac{4N+1}{2\pi}\right)^{1/4} \cdot H_{2N}\left(\sqrt{\frac{4N+1}{2}}\frac{r}{s}\right)\exp\left(-\frac{4N+1}{4}\left(\frac{r}{s}\right)^2\right)$	$\frac{1}{2}$
	2	$A\sqrt{2N+1}\,L_N\left((2N+1)\left(\frac{r}{s}\right)^2\right) \cdot \exp\left(-\frac{2N+1}{2}\left(\frac{r}{s}\right)^2\right)$	1
	3	$\frac{(-1)^N}{\sqrt{3}}\frac{s}{r}\left[\frac{A}{2^N\sqrt{(2N+1)!}}\left(\frac{4N+3}{2\pi}\right)^{1/4} \cdot H_{2N+1}\left(\sqrt{\frac{4N+3}{2}}\frac{r}{s}\right)\exp\left(-\frac{4N+3}{4}\left(\frac{r}{s}\right)^2\right)\right]$	$\frac{3}{2}$
Time		$\frac{A}{2^N\sqrt{(2N+1)!}}\left(\frac{4N+3}{2\pi}\right)^{1/4} \cdot H_{2N+1}\left(\sqrt{\frac{4N+3}{2}}\frac{t}{s}\right)\exp\left(-\frac{4N+3}{4}\left(\frac{t}{s}\right)^2\right)$	$\frac{3}{2}$

* Stationary functions become the optimum impulse responses for $N = 0$.

TABLE II

FOURIER TRANSFORMS OF THE STATIONARY FUNCTIONS AND UNCERTAINTIES IN THE CASE OF "INFINITE AREA"

Dimension		Fourier transforms $\phi_N(\rho), \phi_N(\omega)$; $\rho = \|\xi\| \geqq 0$, $-\infty < \omega < \infty$, $\nu = \frac{n-2}{2}$, $N = 0, 1, 2, \cdots$ *	Uncertainties $s\sigma \geqq n/2$
Space	n	$A \sqrt{\frac{n}{\Omega_n} \frac{N!\Gamma(\nu+1)}{\Gamma(N+\nu+1)}} \left(\frac{2}{(4N+n)\pi}\right)^{n/4} \cdot \exp\left(-\frac{1}{2}\frac{2}{4N+n}(s\rho)^2\right) \sum_{k=0}^{N} (-2)^k \binom{N+\nu}{N-k} \cdot L_k^{(\nu)}\left(\frac{1}{2}\frac{2}{4N+n}(s\rho)^2\right)$	$\Delta_1 = \frac{n}{2}$
	1	$\frac{A2^N N!}{\sqrt{2(2N)!}} \left(\frac{2}{(4N+1)\pi}\right)^{1/4} \exp\left(-\frac{(s\rho)^2}{4N+1}\right) \cdot \sum_{k=0}^{N} \frac{1}{2^k k!} \binom{N-\frac{1}{2}}{N-k} H_{2k}\left(\frac{s\rho}{\sqrt{4N+1}}\right)$	$\frac{1}{2}$
	2	$\frac{A}{\pi\sqrt{2N+1}} \exp\left(-\frac{(s\rho)^2}{4N+2}\right) \cdot \sum_{k=0}^{N} (-2)^k \binom{N}{N-k} L_k\left(\frac{(s\rho)^2}{4N+2}\right)$	1
	3	$\frac{A2^N N!\sqrt{3}}{\pi\sqrt{2\cdot(2N+1)!}} \left(\frac{2}{(4N+3)\pi}\right)^{1/4} \frac{1}{s\rho}$ $\exp\left(-\frac{(s\rho)^2}{4N+3}\right) \sum_{k=0}^{N} \frac{1}{2^{k+1}k!} \binom{N+\frac{1}{2}}{N-k} H_{2k+1}\left(\frac{s\rho}{\sqrt{4N+3}}\right)$	$\frac{3}{2}$
Time			$\frac{3}{2}$

* Fourier transforms become the optimum transfer function for $N = 0$.

spatial filters coincide with those of temporal filters except for the term $(-1)^N s/\sqrt{3}\, r$. Their uncertainties coincide with each other entirely at the value of $\frac{3}{2}$. It seems that the above phenomena might be a consequence of the fact that the actual world is constructed from the "3-dimensional" space and the "1-dimensional" time. We expect that the future studies will solve this problem.

TABLE III

OPTIMUM CHARACTERISTICS AND UNCERTAINTIES IN THE CASE OF "INFINITE AREA"

Dimension		Optimum impulse responses $f_0(r)$, $f_0(t)$ $r = \|\mathbf{x}\| \geqq 0, t \geqq 0$	Optimum transfer functions $\phi_0(\rho)$, $\phi_0(\omega)$ $\rho = \|\xi\| \geqq 0$, $-\infty < \omega < \infty$	Uncertainties $s\sigma \geqq \frac{n}{2}$
Space	n	$A\sqrt{\frac{\Omega_n}{n}}\left(\frac{n}{2\pi}\right)^{n/4} \cdot \exp\left(-\frac{1}{2}\frac{n}{2}\left(\frac{r}{s}\right)^2\right)$	$A\sqrt{\frac{n}{\Omega_n}}\left(\frac{2}{n\pi}\right)^{n/4} \cdot \exp\left(-\frac{1}{2}\frac{2}{n}(s\rho)^2\right)$	$\Delta_1 = \frac{n}{2}$
	1	$A\left(\frac{2}{\pi}\right)^{1/4} \cdot \exp\left(-\frac{1}{4}\left(\frac{r}{s}\right)^2\right)$	$A\left(\frac{1}{2\pi}\right)^{1/4} \exp\left(-(s\rho)^2\right)$	$\frac{1}{2}$
	2	$A \exp\left(-\frac{1}{2}\left(\frac{r}{s}\right)^2\right)$	$\frac{A}{\pi} \exp\left(-\frac{(s\rho)^2}{2}\right)$	1
	3	$A\left(\frac{6}{\pi}\right)^{1/4} \cdot \exp\left(-\frac{3}{4}\left(\frac{r}{s}\right)^2\right)$	$\frac{A}{\pi}\left(\frac{1}{6\pi}\right)^{1/4} \exp\left(-\frac{(s\rho)^2}{3}\right)$	$\frac{3}{2}$
Time		$\sqrt{3}\,\frac{t}{s}\left[A\left(\frac{6}{\pi}\right)^{1/4} \cdot \exp\left(-\frac{3}{4}\left(\frac{t}{s}\right)^2\right)\right]$	$2A\left(\frac{2}{3\pi}\right)^{1/4}\left[1 - \frac{2s\omega}{3}F\left(\frac{s\omega}{\sqrt{3}}\right) - i\sqrt{\frac{\pi}{3}}\, s\omega \exp\left(-\frac{(s\omega)^2}{3}\right)\right]^*$	$\frac{3}{2}$

* The function $F(x) = e^{-x^2} \int_0^x e^{t^2}\, dt$ is the Dawson's integral (Abramowitz and Stegun, Ed., 1964, p. 319).

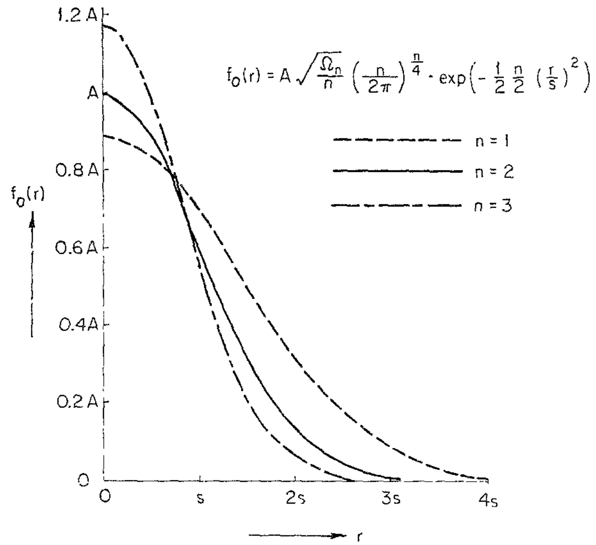

FIG. 1. Variation of the impulse response of the optimum n-dimensional spatial filter in the case of "infinite area" along a radial line (n = 1–3).

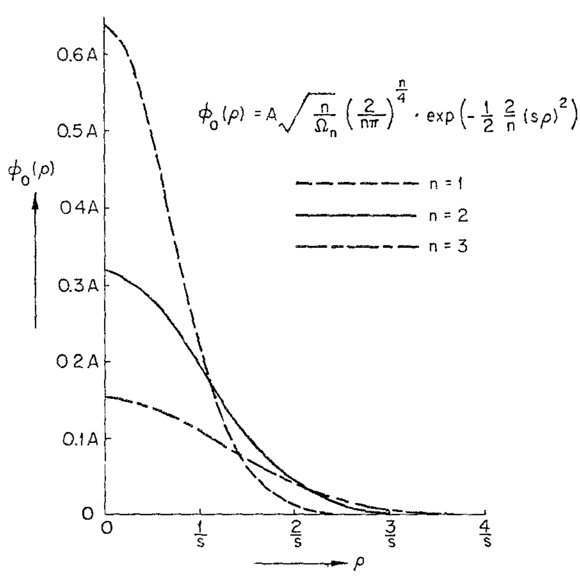

FIG. 2. Variation of the transfer function of the optimum n-dimensional spatial filter in the case of "infinite area" along a radial line (n = 1–3).

In this way, the optimum characteristics of the low-pass spatial filters have been obtained and the uncertainty relation $s\sigma \geqq n/2$ has been shown. However, the equality sign in this inequality cannot be achieved in the case that the spatial domain or the frequency domain are restricted. Such cases will be considered in the following section.

IV. THE CASE OF SPACE LIMITATION

In this case, $f(\mathbf{x})$ vanishes outside the hypersphere $S_n(R)$ of radius R with the center at the origin of E^n. Hence, it is convenient to define the spread of $f(\mathbf{x})$ by the radius R of $S_n(R)$ instead of the definition given by Eq. (12) in Section III (Hosono and Oowaku, 1965). Since $\phi(\xi)$ is allowed to spread over entire $\mathcal{E}^n$ in this case, the spread of $\phi(\xi)$, denoted by σ, is defined in the same way as in Section III. Then the problem is formulated as that of minimizing the following equation (51) under subsidiary condition (52):

$$\sigma^2 = J[f] = \frac{\int_{\mathcal{E}^n} \| \xi \|^2 \{\phi(\xi)\}^2 \, d\xi}{\int_{\mathcal{E}^n} \{\phi(\xi)\}^2 \, d\xi} \tag{51}$$

condition:

$$\frac{1}{V_n(R)} \int_{Sn(R)} \{f(\mathbf{x})\}^2 \, d\mathbf{x} = A^2 \tag{52}$$

Rewriting this problem in the term of $f(r)$, we have

$$\sigma^2 = J[f] = \frac{n}{A^2 R^n} \int_0^R r^{n-1} \{f'(r)\}^2 \, dr \tag{53}$$

conditions:

$$\begin{cases} \dfrac{n}{A^2 R^n} \displaystyle\int_0^R r^{n-1} \{f(r)\}^2 \, dr = 1 & (54) \\ f'(0) = 0, f(R) = 0 & (55) \end{cases}$$

The former of the boundary conditions (55) holds by the same reason as in previous section. The latter is deduced from the assumption (vi).

Calculating the variation of $J[f]$ under subsidiary condition (54), we have

$$\delta J = \frac{2n}{A^2 R^n} \int_0^R r^{n-1} f'(r) \delta f'(r) \, dr + \lambda \frac{2n}{A^2 R^n} \int_0^R r^{n-1} f(r) \delta f(r) \, dr, \tag{56}$$

where λ is Lagrange's multiplier. Since $r^{n-1}f'(r)$ and $\delta f(r)$ are absolutely continuous in the interval $[0, R]$, we can integrate by parts the first term on the right-hand side of Eq. (56), and hence

$$\delta J = \frac{2n}{A^2R^n}\left[r^{n-1}f'(r)\delta f(r) \Big|_0^R \right. - \int_0^R \{r^{n-1}f''(r) + (n-1)r^{n-2}f'(r)\}\delta f(r)\, dr + \lambda \int_0^R r^{n-1}f(r)\delta f(r)\, dr. \tag{57}$$

Now, $\delta f(r)$ is bounded in $[0, R]$ because of its absolute continuity in $[0, R]$, and hence, by Eq. (55), we have

$$r^{n-1}f'(r)\delta f(r)|_{r=o} = 0. \tag{58}$$

$r^{n-1}f'(r)$ is also bounded in $[0, R]$ because of its absolute continuity in $[0, R]$, and hence, by Eq. (55), we have

$$r^{n-1}f'(r)\delta f(r)|_{r=R} = 0. \tag{59}$$

From Eqs. (58) and (59), the first term on the right-hand side of Eq. (57) becomes zero, so that

$$\delta J = (-)\frac{2n}{A^2R^n}\int_0^R \{rf''(r) + (n-1)f'(r) - \lambda rf(r)\}r^{n-2}\delta f(r)\, dr = 0. \tag{60}$$

Since Eq. (60) must be valid for any arbitrary function $\delta f(r)$ which satisfies assumptions (i)–(vii) in Section II, we have the following Euler's differential equation

$$rf''(r) + (n-1)f'(r) - \lambda rf(r) = 0. \tag{61}$$

We can easily prove the following two facts. First, in the case of $\lambda > 0$, if we make the substitution

$$f(r) = r^{-\nu}u(\sqrt{\lambda}r)$$

in Eq. (61), then we have the modified Bessel's differential equation. Hence Eq. (61) has no soluton which satisfies the boundary conditions (55) (See Appendix B). Second, in the case of $\lambda = 0$, Eq. (61) reduces to the form

$$rf''(r) + (n-1)f'(r) = 0,$$

so that Eq. (61) has no solution which satisfies the boundary conditions (55) (See Appendix C).

Consequently, attention is directed to the case of $\lambda < 0$.

If in Eq. (61) we make the substitutions

$$\begin{cases} f(r) = r^{-\nu} u(\beta r), \\ \beta = \sqrt{-\lambda} > 0, \qquad \nu = (n-2)/2, \\ t = \beta r \geqq 0, \end{cases} \tag{62}$$

then we find that Eq. (61) becomes

$$u''(t) + \frac{1}{t} u'(t) + \left(1 - \frac{\nu^2}{t^2}\right) u(t) = 0. \tag{63}$$

Since this is the Bessel's differential equation of order ν, its general solutions are of the forms

$$u(t) = C_1 J_\nu(t) + C_2 N_\nu(t), \tag{64}$$

where $J_\nu(t)$ and $N_\nu(t)$ are the Bessel's and Neumann's functions of order ν, respectively, and C_1 and C_2 are any arbitrary real constants. By Eqs. (64) and (62), the solutions of Eq. (61) are of the forms

$$f(r) = C_1 r^{-\nu} J_\nu(\beta r) + C_2 r^{-\nu} N_\nu(\beta r). \tag{65}$$

In order to obtain $C_2 = 0$ from the former of the boundary conditions (55), we observe first

$$r^{-\nu} J_\nu(\beta r) = \left(\frac{\beta}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu + k + 1)} \left(\frac{\beta r}{2}\right)^{2k} \tag{66}$$

(Watson, 1952, p. 40). When we differentiate the series, which converges uniformly, on the right-hand side of this equation term-by-term, and put $r = 0$, we have

$$\frac{d}{dr} \{ r^{-\nu} J_\nu(\beta r) \} \Big|_{r=0} = 0. \tag{67}$$

In the case of $\nu = -\frac{1}{2}$, it follows that

$$\begin{aligned} \sqrt{r}\, N_{-1/2}(\beta r) &= \sqrt{r}\, J_{1/2}(\beta r) \\ &= \sqrt{\frac{2}{\pi\beta}} \sin \beta r \end{aligned}$$

(Abramowitz and Stegun, Ed., 1964, pp. 438–439), and hence

$$\frac{d}{dr}\{\sqrt{\lambda}\,N_{-1/2}(\beta r)\}\Big|_{r=0} = \sqrt{\frac{2\beta}{\pi}} > 0. \tag{68}$$

In the case of $\nu = 0$, it follows that

$$N_0(\beta r) \sim \frac{2}{\pi}\log \beta r \qquad (r \to 0)$$

(Abramowitz and Stegun, Ed., 1964, p. 360), so that

$$\frac{d}{dr} N_0\,(\beta r) \sim \frac{2}{\pi r} \to \infty \;\; (r \to 0). \tag{69}$$

In the case of $\nu > 0$, it follows that

$$r^{-\nu}N_\nu(\beta r) \sim (-)\,\frac{2^\nu\Gamma(\nu)}{\pi\beta^\nu r^{2\nu}} \qquad (r \to 0)$$

(Abramowitz and Stegun, Ed., 1964, p. 360), so that

$$\frac{d}{dr}\{r^{-\nu}N_\nu(\beta r)\} \sim \frac{2^{\nu+1}\Gamma(\nu+1)}{\pi\beta^\nu r^{2\nu+1}} \to \infty \qquad (r \to 0). \tag{70}$$

From Eqs. (67) $\sim$ (70), C_2 must be zero in order that $f'(0) = 0$. Hence, Eq. (65) deduces to

$$f(r) = C_1 r^{-\nu} J_\nu(\beta r). \tag{71}$$

Then, in order that $f(R) = 0$ it must follow that

$$\beta = \frac{j_{\nu N}}{R}\;(N = 1, 2, 3, \cdots), \tag{72}$$

where $j_{\nu 1}, j_{\nu 2}, \cdots, j_{\nu N}, \cdots$ are the positive zeros of $J_\nu(z)$ arranged in ascending order of magnitude. If we denote the stationary functions corresponding to $\beta = j_{\nu N}/R$ by $f_N(r)$, then we have

$$f_N(r) = C_1 r^{-\nu} J_\nu\left(j_{\nu N}\frac{r}{R}\right), \qquad 0 \leqq r \leqq R, \quad N = 1, 2, 3, \cdots. \tag{73}$$

In order to obtain C_1 from Eqs. (54) and (73), we observe first

$$\int^z z\{J_\nu(\alpha z)\}^2\,dz = \frac{z^2}{2}\left[\{J_\nu(\alpha z)\}^2 - J_{\nu-1}(\alpha z)J_{\nu+1}(\alpha z)\right], \tag{74}$$

(Watson, 1952, p. 135) and the recurrence formula

$$J_{\nu-1}(z) = 2\nu z^{-1} J_\nu(z) - J_{\nu+1}(z) \tag{75}$$

(Watson, 1952, p. 45). Substituting Eq. (75) into Eq. (74), we have

$$\int^z z\{J_\nu(\alpha z)\}^2\, dz = \frac{z^2}{2}\left[\{J_\nu(\alpha z)\}^2 + \{J_{\nu+1}(\alpha z)\}^2 - \frac{2\nu}{\alpha z} J_\nu(\alpha z) J_{\nu+1}(\alpha z)\right]. \tag{76}$$

Substituting Eq. (73) into Eq. (54) and using Eq. (76), we have

$$\frac{A^2 R^n}{n} = C_1^2 R^2 \int_0^1 r\{J_\nu(j_{\nu N} r)\}^2\, dr = \frac{C_1^2 R^2}{2}\{J_{\nu+1}(j_{\nu N})\}^2,$$

so that

$$C_1 = \pm \frac{(-1)^{N-1} A}{j_{\nu+1}(j_{\nu N})}\sqrt{\frac{2}{n}}\, R^\nu. \tag{77}$$

It is not essential in these problems whether C_1 is positive or negative, and hence the positive sign is accepted in this paper. Substituting Eq. (77) into Eq. (73), we have

$$f_N(r) = \frac{(-1)^{N-1} A}{J_{\nu+1}(j_{\nu N})}\sqrt{\frac{2}{n}}\left(\frac{R}{r}\right)^\nu J_\nu\left(J_{\nu N}\frac{r}{R}\right), \quad 0 \leqq r \leqq R, \quad N = 1, 2, 3, \cdots. \tag{78}$$

These functions satisfy all assumptions in Section II certainly.

Multiplying Eq. (61) by $(n/A^2 R^n) r^{n-2} f(r)$ and integrating from 0 to R, we have

$$\frac{n}{A^2 R^n}\int_0^R \left[\frac{d}{dr}\{r^{n-1} f'(r) f(r)\} - r^{n-1}\{f'(r)\}^2 - \lambda r^{n-1}\{f(r)\}^2\right] dr = 0.$$

From Eqs. (53) and (54), it follows that

$$J[f] = -\lambda + \frac{n}{A^2 R^n}\Big[r^{n-1} f'(r) f(r)\Big]_0^R, \tag{79}$$

because of the absolute continuity of $r^{n-1}f'(r)f(r)$ in $[0, R]$. The second term on the right-hand side of Eq. (79) becomes zero by the reasoning used at Eqs. (58) and (59), so that

$$J[f] = -\lambda. \tag{80}$$

By Eqs. (62), (72) and (80), it follows that

$$\sigma_N{}^2 = J[f_N] = \left(\frac{j_{\nu N}}{R}\right)^2, \qquad N = 1, 2, 3, \cdots, \tag{81}$$

and hence

$$\sigma^2 = J[f] \geqq \left(\frac{j_\nu}{R}\right)^2, \tag{82}$$

where $j_\nu \equiv j_{\nu 1}$ is the minimum positive zero of $J_\nu(z)$. Consequently, the following uncertainty relation holds:

$$R\sigma \geqq j_\nu . \tag{83}$$

We shall call $\Delta_2 \equiv j_\nu$ "the uncertainty of the family of filters" in the case of space limitation.

Since the equality sign in Eq. (83) holds only for $N = 1$, putting $N = 1$ in Eq. (78), we find that the optimum characteristic of the spatial filter in the case of space limitation is as follows:

$$\begin{gathered} f_1(r) = \frac{A}{J_{\nu+1}(j_\nu)} \sqrt{\frac{2}{n}} \left(\frac{R}{r}\right)^\nu J_\nu\left(j_\nu \frac{r}{R}\right), \\ r = \|\mathbf{x}\|, 0 \leqq r \leqq R, \nu = (n-2)/2. \end{gathered} \tag{84}$$

In Appendix D, we shall give the transfer function $\phi_N(\rho)$ corresponding to $f_N(r)$ and also we shall give the relation between $f_N(0)$ and $\phi_N(j_{\nu N}/R)$, that is,

$$f_N(0)\phi_N\left(\frac{j_{\nu N}}{R}\right) = \frac{A^2}{(2\pi)^{\nu+1}} \tag{85}$$

where $\sigma_N = j_{\nu N}/R$ is the square root of the centered second moment of $\{\phi_N(\rho)\}^2$ (cf. Eq. (81).

Putting $N = 1$ in Eqs. (D.13) $\sim$ (D.15), we find that the optimum transfer function $\phi_1(\rho)$ of the spatial filter in the case of space limitation is as follows,

$$\phi_1(\rho) = \frac{A j_\nu \sqrt{2n}}{\Omega_n} \frac{J_\nu(R\rho)}{(R\rho)^\nu[(j_\nu)^2 - (R\rho)^2]} \tag{86}$$

$$= \begin{cases} \dfrac{A\sqrt{2n}}{j_\nu(2\pi)^{\nu+1}} & :\rho = 0 \quad (87) \\[2ex] A\sqrt{\dfrac{n}{2}} \dfrac{J_{\nu+1}(j_\nu)}{\Omega_n(j_\nu)^\nu} = \dfrac{A^2}{(2\pi)^{\nu+1}} \dfrac{1}{f_1(0)} & \quad (88) \\[2ex] & :\rho = \dfrac{j_\nu}{R} \\[2ex] 0 & :\rho = \dfrac{j_{\nu k}}{R} \ (k = 2, 3, 4, \cdots) \quad (89) \\[2ex] \dfrac{Aj_\nu \sqrt{2n}}{\Omega_n} \dfrac{J_\nu(R\rho)}{(R\rho)^\nu[(j_\nu)^2 - (R\rho)^2]} & \\[2ex] & :\rho \fallingdotseq\!\!\!\!/\ \dfrac{j_{\nu k}}{R} \ (k = 0, 1, 2, \cdots) \quad (90) \end{cases}$$

$$\sim \frac{(-)2\sqrt{n}A_{j\nu}}{\sqrt{\pi}\,\Omega_n} \frac{(R\rho - (n-1)\pi/4)}{(R\rho)^{\nu+5/2}} \qquad :\rho \to \infty \tag{91}$$

$$\rho = \|\xi\| \geqq 0, \quad \nu = (n-2)/2.$$

By Eq. (89), zeros of $\phi_1(\rho)$ of the 1-dimensional and 3-dimensional spatial filters are

$$\rho_k = \frac{(2k+1)\pi}{2R} \quad \text{and} \quad \rho_k = \frac{(k+1)\pi}{R} \qquad (k = 1, 2, 3, \cdots), \tag{92}$$

respectively, so that all of the intervals between ρ_k and ρ_{k+1} have the same value π/R.

The definition of the spread of $f(\mathbf{x})$ in this section differs from the definition in Section III. However, it is interesting to compare the uncertainty Δ_2 obtained above with the uncertainty Δ_1 obtained in Section III. If we calculate the spread of $f_1(\mathbf{x})$, denoted by s_1, according to the definition in Section III (cf. Eq. (16)), then we find that the uncertainty of the optimum filter in the case of space limitation is as follows,

$$\Delta_3 \equiv s_1\sigma_1 = \sqrt{\frac{(j_\nu)^2 + 2(\nu^2 - 1)}{3}} \tag{93}$$

TABLE IV
UNCERTAINTIES OF SPATIAL FILTERS

Dimension n	$\Delta_1 = \frac{n}{2}$	$\Delta_2 = j_\nu$	$\Delta_3 = \sqrt{\frac{(j_\nu)^2 + 2(\nu^2 - 1)}{3}}$	$\frac{\Delta_3}{\Delta_1}$
1	0.5	1.5708	0.5679	1.1357
2	1.0	2.4048	1.1230	1.1223
3	1.5	3.1416	1.6703	1.1135

(See Appendix E). It follows certainly that

$$\Delta_1 < \Delta_3 < \Delta_2 .$$

However, by means of the asymptotic formula

$$j_\nu \sim \nu + 1.855757\nu^{1/3} + O(\nu^{-1/3}) \quad (\nu \to \infty) \tag{94}$$

(Watson, 1952, p. 521), these three uncertainties Δ_1, Δ_2, and Δ_3 coincide with each other asymptotically as n tends to infinity. Table IV gives these three uncertainties and Δ_3/Δ_1 for $n = 1 \sim 3$.

The differences between the optimum characteristics of spatial filters and those of the temporal filter which have been already obtained by Hosono and Oowaku (1965)[6] are given in Tables V and VI. In constructing these tables, we used the following formulae:

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z \tag{95}$$

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z \tag{96}$$

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right) \tag{97}$$

(Watson, 1952, p. 54).

The structure of $\phi_1(\rho)$ for arbitrary n is illustrated in Fig. 3. Variations of $f_1(r)$ and $\phi_1(\rho)$ are illustrated for $n = 1 \sim 3$ in Figs. 4 and 5, respectively.

[6] If we replace x by $x + R$ in the optimum characteristics of the 1-dimensional spatial filter whose impulse response is restricted to $| x | \leqq R$, then we can obtain the optimum characteristics of the temporal filter whose impulse response is restricted to $0 \leqq t \leqq 2R$. The spreads of the former and the latter are R and $2R$, respectively, since the spread of the impulse response of the temporal filter whose impulse response is restricted to $0 \leqq t \leqq T$ is defined by T.

Considering that filters are isotropic, let us pick up only even functions from the stationary functions of 1-dimensional spatial filters which had been obtained by Hosono and Oowaku (1965), and let us renumber them. Then we find that they coincide with the stationary functions of the 1-dimensional spatial filters in Table V.

The stationary functions of the variational problem for 3-dimensional spatial filters coincide with those of temporal filters except for the term

TABLE V

STATIONARY FUNCTIONS, THEIR FOURIER TRANSFORMS, AND UNCERTAINTIES IN THE CASE OF "SPACE LIMITATION"

Dimension		$f_N(r), f_N(t)$; $N = 1, 2, 3, \cdots$ * $r = \|\mathbf{x}\|$, $0 \leqq r \leqq R$, $0 \leqq t \leqq R$, $\nu = \frac{n-2}{2}$	$\phi_N(\rho), \phi_N(\omega)$; $N = 1, 2, 3, \cdots$ * $\rho = \|\xi\| \geqq 0$, $-\infty < \omega < \infty$, $\nu = \frac{n-2}{2}$	Uncertainties: $R\sigma \geqq j_\nu$	Uncertainties: $s_1\sigma_1 = \sqrt{\frac{(j_\nu)^2 + 2(\nu^2-1)}{3}}$
Space	n	$\frac{(-1)^{N-1}A}{j_{\nu+1}(j_{\nu N})} \sqrt{\frac{2}{n}} \left(\frac{R}{r}\right)^\nu \cdot J_\nu\left(j_{\nu N}\frac{r}{R}\right)$	$\frac{(-1)^{N-1}A j_{\nu N}\sqrt{2n}}{\Omega_n} \cdot \frac{J_\nu(R\rho)}{(R\rho)^\nu[(j_{\nu N})^2 - (R\rho)^2]}$	$\Delta_2 = j_\nu$	$\Delta_3 = \sqrt{\frac{(j_\nu)^2 + 2(\nu^2-1)}{3}}$
	1	$A\sqrt{2} \cdot \cos\left(\frac{(2N-1)\pi}{2}\frac{r}{R}\right)$	$\frac{(-1)^{N-1}2(2N-1) \cdot \sqrt{\pi}A\cos(R\rho)}{\{(2N-1)\pi\}^2 - (2R\rho)^2}$	$\frac{\pi}{2}$	$\sqrt{\frac{\pi^2-6}{12}}$
	2	$\frac{(-1)^{N-1}A}{J_1(j_{0N})} J_0\left(j_{0N}\frac{r}{R}\right)$	$\frac{(-1)^{N-1}A\, j_{0N} J_0(R\rho)}{\pi[(j_{0N})^2 - (R\rho)^2]}$	j_0	$\sqrt{\frac{(j_0)^2-2}{3}}$
	3	$\frac{R}{\sqrt{3}\,r}\left\{A\sqrt{2} \cdot \sin\left(N\pi\frac{r}{R}\right)\right\}$	$\frac{(-1)^{N-1}AN\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{\sin(R\rho)}{(R\rho)[(N\pi)^2 - (R\rho)^2]}$	π	$\sqrt{\frac{2\pi^2-3}{6}}$
Time		$A\sqrt{2}\sin\left(N\pi\frac{t}{R}\right)$	$\frac{AN\sqrt{\pi}[1 + (-1)^{N-1} \cdot \exp(-iR\omega)]}{(N\pi)^2 - (R\omega)^2}$	π	$\sqrt{\frac{2\pi^2-3}{6}}$

* The functions f_N and ϕ_N become the optimum characteristics for $N = 1$.

$R/\sqrt{3}\, r$. Their uncertainties coincide with each other entirely at the value of π. The same phenomena have been found in the case of infinite area.

If we interchange $\mathbf{x}$ and $\boldsymbol{\xi}$ in the results obtained above, then we have the solutions for the case of "frequency limitation". That is, in the case that $\phi(\boldsymbol{\xi})$ vanishes outside the hypersphere $\Sigma_n(P)$ of radius P with the center at the origin of $\mathcal{E}^n$, the spread of $\phi(\boldsymbol{\xi})$ is defined by the radius P of $\Sigma_n(P)$. The spread of $f(\mathbf{x})$, denoted by s, is defined by Eq. (12) in

TABLE VI

OPTIMUM CHARACTERISTICS AND UNCERTAINTIES IN THE CASE OF "SPACE LIMITATION"

Dimension		Optimum impulse response $f_1(r), f_1(t)$ $r = \lVert\mathbf{x}\rVert, 0 \leqq r \leqq R,$ $0 \leqq t \leqq R$	Optimum transfer function $\phi_1(\rho), \phi_1(\omega)$ $\rho = \lVert\boldsymbol{\xi}\rVert \geqq 0,$ $-\infty < \omega < \infty$	Uncertainties $R\sigma \geqq j_\nu$	Uncertainties $s_1\sigma_1 = \sqrt{\dfrac{(j_\nu)^2 + 2(\nu^2-1)}{3}}$
Space	n	$\dfrac{A}{J_{\nu+1}(j_\nu)} \sqrt{\dfrac{2}{n}} \left(\dfrac{R}{r}\right)^\nu \cdot J_\nu\left(j_\nu \dfrac{r}{R}\right)$	$\dfrac{A j_\nu \sqrt{2n}}{\Omega_n} \cdot \dfrac{J_\nu(R\rho)}{(R\rho)^\nu[(j_\nu)^2 - (R\rho)^2]}$	$\Delta_2 = j_\nu$	$\Delta_3 = \sqrt{\dfrac{(j_\nu)^2 + 2(\nu^2-1)}{3}}$
Space	1	$A\sqrt{2}\cos\left(\dfrac{\pi}{2}\dfrac{r}{R}\right)$	$\dfrac{2A\sqrt{\pi}\cos(R\rho)}{\pi^2 - (2R\rho)^2}$	$\dfrac{\pi}{2}$	$\sqrt{\dfrac{\pi^2-6}{12}}$
Space	2	$\dfrac{A}{J_1(j_0)} J_0\left(j_0 \dfrac{r}{R}\right)$	$\dfrac{A\, j_0 J_0(R\rho)}{\pi[(j_0)^2 - (R\rho)^2]}$	j_0	$\sqrt{\dfrac{(j_0)^2 - 2}{3}}$
Space	3	$\dfrac{R}{\sqrt{3}\, r}\left\{A\sqrt{2} \cdot \sin\left(\pi \dfrac{r}{R}\right)\right\}$	$\dfrac{A\sqrt{3}\sin(R\rho)}{2\sqrt{\pi}\,(R\rho)[\pi^2 - (R\rho)^2]}$	π	$\sqrt{\dfrac{2\pi^2-3}{6}}$
Time		$A\sqrt{2}\sin\left(\pi \dfrac{t}{R}\right)$	$\dfrac{A\sqrt{\pi}[1+\exp(-iR\omega)]}{\pi^2 - (R\omega)^2}$	π	$\sqrt{\dfrac{2\pi^2-3}{6}}$

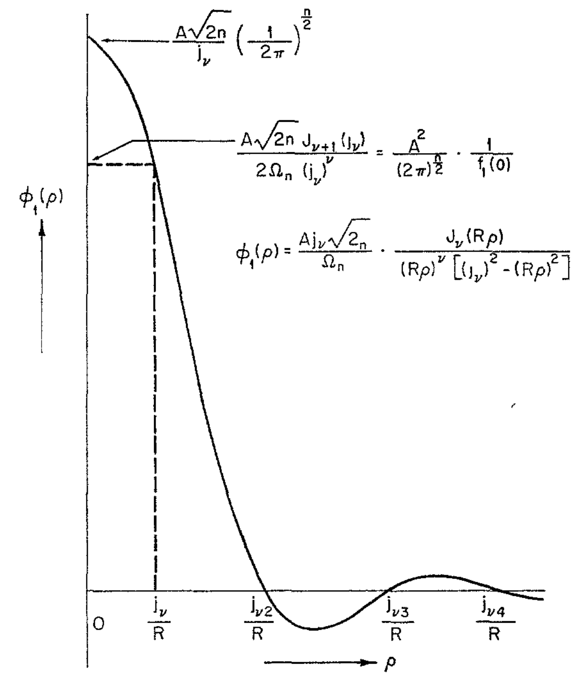

FIG. 3. Structure of the transfer function of the optimum n-dimensional spatial filter under "space limitation."

Section III. Hence, the problem is formulated as that of minimizing the following equation (98) under subsidiary condition (99):

$$s^2 = J[\phi] = \frac{\int_{E^n} \|\mathbf{x}\|^2 \{f(\mathbf{x})\}^2 \, d\mathbf{x}}{\int_{E^n} \{f(\mathbf{x})\}^2 \, d\mathbf{x}} \tag{98}$$

condition:

$$\frac{1}{V_n(P)} \int_{\Sigma_n(P)} \{\phi(\xi)\}^2 \, d\xi = A^2 \tag{99}$$

In mathematical forms this problem is similar to the problem in the case of space limitation, and hence the uncertainty relation and the optimum characteristics in the case of frequency limitation are of the

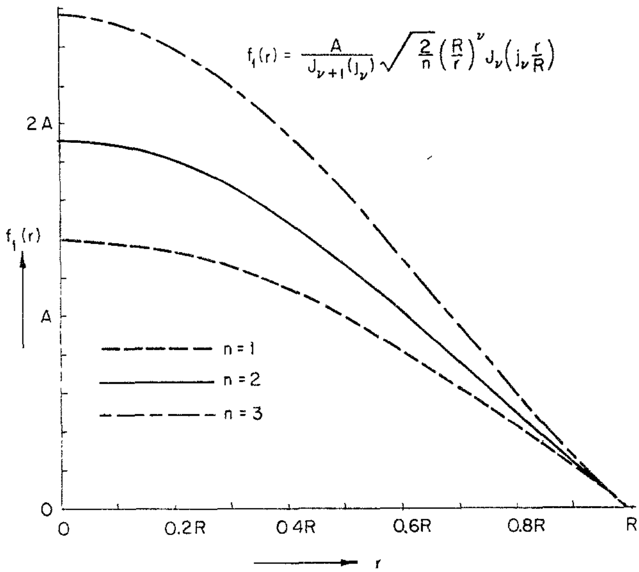

FIG. 4. Variation of the impulse response of the optimum n-dimensional spatial filter under "space limitation" along a radial line (n = 1–3).

forms

$$sP \geqq j_\nu, \tag{100}$$

$$f_1(r) = \frac{Aj_\nu\sqrt{2n}}{\Omega^n}\,\frac{J_\nu(Pr)}{(Pr)^\nu[(j_\nu)^2 - (Pr)^2]}, \tag{101}$$

$$\phi_1(\rho) = \frac{A}{J_{\nu+1}(j_\nu)}\sqrt{\frac{2}{n}}\left(\frac{P}{\rho}\right)^\nu J_\nu\left(j_\nu\,\frac{\rho}{P}\right), \tag{102}$$

$$r = \|\mathbf{x}\| \geqq 0, \quad \rho = \|\xi\|, \quad 0 \leqq \rho \leqq P,$$

$$\nu = (n-2)/2.$$

V. CONCLUDING REMARKS

The uncertainty relations between spatial resolution and spatial frequency resolution of the spatial filters have been obtained by means of the variational method. The n-dimensional spatial filters are assumed to be linear, homogeneous, isotropic and low-pass. The optimum filters in the sense of minimizing the product of their resolutions have

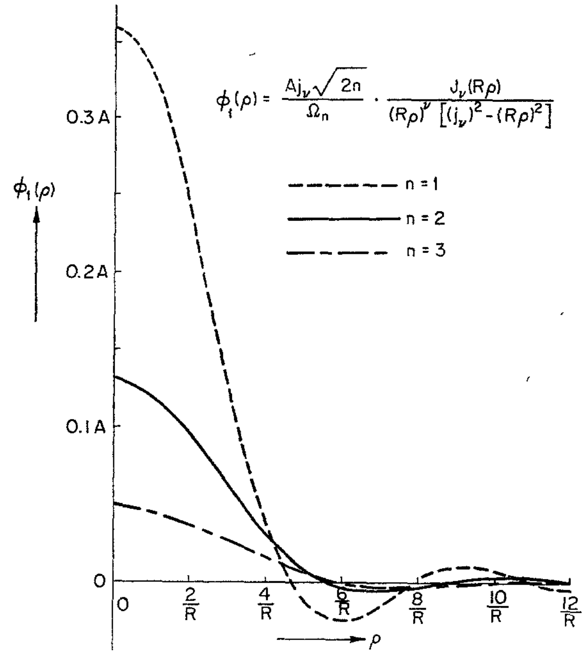

FIG. 5. Variation of the transfer function of the optimum n-dimensional spatial filter under "space limitation" along a radial line (n = 1–3).

been obtained. The following results have been obtained: *1. The case of infinite area.* The problem is formulated by Eqs. (10)–(12). The uncertainty relation is of the form $s\sigma \geqq n/2$ (Eq. (43)). The associated Laguerre's polynomials appear in the stationary functions of the variational problem (Eq. (37)). The optimum characteristics are given by Eqs. (44) and (46). *2. The case of space limitation.* The problem is formulated by Eqs. (51) and (52). The uncertainty relation is of the form $R\sigma \geqq j_\nu$ (Eq. (83)). The Bessel functions appear in the stationary functions (Eq. (78)). The optimum characteristics are given by Eqs. (84) and (86). *3. The case of frequency limitation.* This case is symmetric to the second case, and the problem is formulated by Eqs. (98) and (99). The uncertainty relation is of the form $sP \geqq j_\nu$ (Eq. (100)). The optimum characteristics are given by Eqs. (101) and (102).

From the signal design aspects, these optimum characteristics may be regarded as the optimum n-dimensional spatial pulse shape, which minimizes the product of its width and the spread of its Fourier transform.

We have obtained the result that the uncertainties of the 3-dimensional spatial filters coincide with those of temporal filters at the value of $\frac{3}{2}$ in the case of infinite area and of π in the case of space or frequency limitation, respectively.

The extensions to band-pass filters and tempo-spatial filters remain to be solved.

APPENDIX A: CALCULATION FOR $\phi_N(\rho)$ IN THE CASE OF INFINITE AREA

Substituting Eqs. (37) and (31) into Eq. (6), we have

$$\begin{aligned}
\phi_N(\rho) &= \frac{1}{V_n(s)} \frac{1}{\rho^\nu} A \sqrt{\frac{\Omega_n}{n} \frac{N!\,\Gamma(\nu+1)}{\Gamma(N+\nu+1)}} \left(\frac{4N+n}{2\pi}\right)^{n/4} \\
&\quad \cdot \int_0^\infty r^{\nu+1} L_N^{(\nu)}\left(\frac{4N+n}{2}\left(\frac{r}{s}\right)^2\right) \\
&\quad \cdot \exp\left(-\frac{4N+n}{4}\left(\frac{r}{s}\right)^2\right) J_\nu(\rho r)\, dr \\
&= \frac{A}{s^n \rho^\nu} \sqrt{\frac{n}{\Omega_n} \frac{N!\Gamma(\nu+1)}{\Gamma(N+\nu+1)}} \left(\frac{4N+n}{2\pi}\right)^{n/4} \sum_{k=0}^N \frac{(-1)^k}{k!} \binom{N+\nu}{N-k} \\
&\quad \cdot \left(\frac{4N+n}{2s^2}\right)^k \int_0^\infty r^{\nu+1+2k} \exp\left(-\frac{4N+n}{4s^2} r^2\right) \\
&\quad \cdot J_\nu(\rho r)\, dr.
\end{aligned} \tag{A.1}$$

From the formula

$$\begin{aligned}
&\int_0^\infty t^{\mu-1} \exp(-a^2 t^2) J_\nu(bt)\, dt \\
&= \frac{b^\nu \Gamma\left(\dfrac{\nu+\mu}{2}\right)}{2^{\nu+1} a^{\nu+\mu} \Gamma(\nu+1)} \exp\left(-\frac{b^2}{4a^2}\right) {}_1F_1\left(\frac{\nu-\mu}{2}+1; \nu+1; \frac{b^2}{4a^2}\right)
\end{aligned} \tag{A.2}$$

which is valid for Re $(\nu+\mu) > 0$, where ${}_1F_1$ is the Kummer's confluent hypergeometric function (Watson, 1952, p. 394), Eq. (A.1) becomes

$$\begin{aligned}
\phi_N(\rho) &= A \sqrt{\frac{n}{\Omega_n} \frac{N!\Gamma(\nu+1)}{\Gamma(N+\nu+1)}} \left(\frac{2}{(4N+n)\pi}\right)^{n/4} \\
&\quad \cdot \exp\left(-\frac{(s\rho)^2}{4N+n}\right) \sum_{k=0}^N (-2)^k \binom{N+\nu}{N-k} \frac{\Gamma(\nu+1+k)}{k!\Gamma(\nu+1)} \\
&\quad \cdot {}_1F_1\left(-k; \nu+1; \frac{(s\rho)^2}{4N+n}\right).
\end{aligned} \tag{A.3}$$

From the formula

$$L_N^{(\nu)}(x) = \frac{\Gamma(\nu + N + 1)}{N!\Gamma(\nu + 1)} {}_1F_1(-N; \nu + 1; x) \tag{A.4}$$

(Abramowitz and Stegun, Ed., 1964, p. 509), Fourier transforms of $f_N(r)$ are of the forms

$$\begin{aligned} \phi_N(\rho) = A \sqrt{\frac{n}{\Omega_n} \frac{N!\Gamma(\nu + 1)}{\Gamma(N + \nu + 1)}} \left(\frac{2}{(4N + n)\pi}\right)^{n/4} \\ \cdot \exp\left(-\frac{1}{2} \frac{2}{4N + n} (s\rho)^2\right) \\ \cdot \sum_{k=0}^{N} (-2)^k \binom{N + \nu}{N - k} L_k^{(\nu)} \left(\frac{1}{2} \frac{2}{4N + n} (s\rho)^2\right). \end{aligned} \tag{A.5}$$

APPENDIX B: SOLUTION OF EQUATION (61) FOR $\lambda > 0$

If in Eq. (61) we make the substitutions

$$\begin{cases} f(r) = r^{-\nu} u(\beta r), \\ \quad \beta = \sqrt{\lambda} > 0, \qquad \nu = (n - 2)/2, \\ \quad t = \beta r \geqq 0, \end{cases} \tag{B.1}$$

then we find that Eq. (61) becomes

$$u''(t) + \frac{1}{t} u'(t) - \left(1 + \frac{\nu^2}{t^2}\right) u(t) = 0. \tag{B.2}$$

Since this is the modified Bessel differential equation of order ν, its general solutions are of the forms

$$u(t) = C_1 I_\nu(t) + C_2 K_\nu(t), \tag{B.3}$$

where $I_\nu(t)$ and $K_\nu(t)$ are the modified Bessel functions of the first and second kind of order ν, respectively, that is,

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(iz) : -\pi < \arg z < \pi/2 \tag{B.4}$$

$$= \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k} : -\pi < \arg z < \pi \tag{B.5}$$

(Watson, 1952, p. 77), and C_1 and C_2 are any arbitrary real constants. Then the solutions of Eq. (61) in the case of $\lambda > 0$ are of the forms

$$f(r) = C_1 r^{-\nu} I_\nu(\beta r) + C_2 r^{-\nu} K_\nu(\beta r). \tag{B.6}$$

It is deduced that $C_2 = 0$ from the former of the boundary conditions (55) by the following way. First, by Eq. (B.5)

$$\frac{d}{dr}\{r^{-\nu}I_\nu(\beta r)\}\Big|_{r=0} = 0. \tag{B.7}$$

In the case of $\nu = -\frac{1}{2}$, it follows that

$$K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

(Watson, 1952, p. 80), and hence

$$\frac{d}{dr}\{\sqrt{r}K_{-1/2}(\beta r)\}\Big|_{r=0} = -\sqrt{\frac{\pi\beta}{2}} < 0. \tag{B.8}$$

In the case of $\nu = 0$, it follows that

$$K_0(\beta r) \sim (-)\log(\beta r) \qquad (r \to 0)$$

(Abramowitz and Stegun, Ed., 1964, p. 375), and hence

$$\frac{d}{dr}K_0(\beta r) \sim \frac{-1}{r} \to -\infty \qquad (r \to 0). \tag{B.9}$$

In the case of $\nu > 0$, it follows that

$$r^{-\nu}K_\nu(\beta r) \sim \frac{2^{\nu-1}\Gamma(\nu)}{\beta^\nu r^{2\nu}} \qquad (r \to 0)$$

(Abramowitz and Stegun, Ed., 1964, p. 375), and hence

$$\frac{d}{dr}\{r^{-\nu}K_\nu(\beta r)\} \sim \frac{(-)2^\nu\Gamma(\nu+1)}{\beta^\nu r^{2\nu+1}} \to -\infty \ (r \to 0). \tag{B.10}$$

By Eqs. (B.7)–(B.10), C_2 must be zero in order that $f'(0) = 0$. Consequently, it follows that

$$f(r) = C_1 r^{-\nu} I_\nu(\beta r). \tag{B.11}$$

Since zeros of $J_\nu(z)$ are all real for $\nu > -1$, then by Eq. (B.4) the real zero of $I_\nu(z)$ is only $z = 0$ if any, so that

$$f(R) = C_1 R^{-\nu} I_\nu(\beta R) \neq 0. \tag{B.12}$$

Hence, Eq. (61) has no solutions which satisfy the boundary conditions (55).

APPENDIX C: SOLUTION OF EQUATION (61) FOR $\lambda = 0$

In the case of $\lambda = 0$, Eq. (61) deduces to the form

$$rf''(r) + (n - 1)f'(r) = 0. \tag{C.1}$$

Moreover, in the case of $n = 1$, Eq. (C.1) becomes

$$f''(r) = 0, \tag{C.2}$$

so that

$$f(r) = C_1 r + C_2 . \tag{C.3}$$

Then, by the boundary conditions (55), we find that $C_1 = C_2 = 0$, and hence Eq. (C.2) has only the solution $f(r) \equiv 0$.

In the case of $n = 2$, Eq. (C.1) becomes

$$rf''(r) + f'(r) = 0, \tag{C.4}$$

so that

$$f(r) = C_1 \log r + C_2 . \tag{C.5}$$

Hence, by the boundary conditions (55), we find that Eq. (C.4) has only the solution $f(r) \equiv 0$.

In the case of $n \geqq 3$, general solutions of Eq. (C.1) are of the forms.

$$f(r) = \frac{C_1}{n - 2} \frac{1}{r^{n-2}} + C_2 . \tag{C.6}$$

Hence, by the boundary conditions (55), we also find that Eq. (C.1) has only the solution $f(r) \equiv 0$.

APPENDIX D: CALCULATION FOR $\phi_N(\rho)$ IN THE CASE OF SPACE LIMITATION

Substituting Eq. (78) into Eq. (6), we have

$$\phi_N(\rho) = \frac{(-1)^{N-1} A \sqrt{2n}}{\Omega_n J_{\nu+1}(j_{\nu N})} \frac{1}{(R\rho)^\nu} \int_0^1 r J_\nu(j_{\nu N} r) J_\nu(R\rho r)\, dr, \tag{D.1}$$

so that, in the case of $R\rho \fallingdotseq j_{\nu N}$, we have

$$\phi_N(\rho) = \frac{(-1)^{N-1} A j_{\nu N} \sqrt{2n}}{\Omega_n} \frac{J_\nu(R\rho)}{(R\rho)^\nu[(j_{\nu N})^2 - (R\rho)^2]} \tag{D.2}$$

by means of the formula

$$\int^z zJ_\nu(az)J_\nu(bz)\,dz$$

$$= \frac{z}{a^2 - b^2}[aJ_{\nu+1}(az)J_\nu(bz) - bJ_\nu(az)J_{\nu+1}(bz)], \tag{D.3}$$

which is valid for $a \neq b$ (Watson, 1952, p. 134). In the case of $R\rho = j_{\nu N}$, by Eq. (76) we have

$$\phi_N\left(\frac{j_{\nu N}}{R}\right) = (-1)^{N-1}A\sqrt{\frac{n}{2}}\frac{J_{\nu+1}(j_{\nu N})}{\Omega_n(j_{\nu N})^\nu}. \tag{D.4}$$

From Eq. (D.2) and the recurrence formula

$$zJ_\nu'(z) = \nu J_\nu(z) - zJ_{\nu+1}(z) \tag{D.5}$$

(Watson, 1952, p. 45), we can obtain easily

$$\phi_N\left(\frac{j_{\nu N}}{R}\right) = \lim_{\rho \to j_{\nu N}/R} \phi_N(\rho). \tag{D.6}$$

If we substitute Eq. (66) into Eq. (D.2) in order to obtain the explicit expression for $\phi_N(0)$, then we have

$$\phi_N(\rho) = \frac{(-1)^{N-1}Aj_{\nu N}\sqrt{2n}}{2^\nu\Omega_n[(j_{\nu N})^2 - (R\rho)^2]}\sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{R\rho}{2}\right)^{2k}, \tag{D.7}$$

so that

$$\phi_N(0) = \frac{(-1)^{N-1}A\sqrt{2n}}{j_{\nu N}(2\pi)^{\nu+1}}. \tag{D.8}$$

From Eqs. (D.2), (D.6) and (D.8), it is found that Eq. (D.2) is valid for every $\rho \geqq 0$. From Eqs. (D.2), (D.4), and (D.8), zeros of $\phi_N(\rho)$ are $j_{\nu k}/R$, where $k = 1, 2, 3, \cdots, N-1, N+1, \cdots$.

By the asymptotic formula of $J_\nu(z)$

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{(2\nu+1)\pi}{4}\right), \tag{D.9}$$

which is valid for large values of $|z|$ provided that $|\arg z| < \pi$ (Watson, 1952, p. 199), the asymptotic formula of $\phi_N(\rho)$ are of the forms

$$\phi_N(\rho) \sim \frac{(-1)^N 2\sqrt{n}Aj_{\nu N}}{\sqrt{\pi}\Omega_n}\frac{\cos(R\rho - (n-1)\pi/4)}{(R\rho)^{\nu+5/2}} \cdot (\rho \to \infty). \tag{D.10}$$

Substituting Eq. (66) into Eq. (78), we have

$$f_N(r) = \frac{(-1)^{N-1}A}{J_{\nu+1}(j_{\nu N})} \sqrt{\frac{2}{n}} \left(\frac{j_{\nu N}}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{j_{\nu N}}{2}\frac{r}{R}\right)^{2k},$$

and hence, by Eq. (D.4) it follows that

$$\begin{aligned} f_N(0) &= \frac{(-1)^{N-1}A}{J_{\nu+1}(j_{\nu N})} \sqrt{\frac{2}{n}} \left(\frac{j_{\nu N}}{2}\right)^{\nu} \frac{1}{\Gamma(\nu+1)} \\ &= \frac{(-1)^{N-1}}{A} \sqrt{\frac{2}{n}} \frac{\Omega_n (j_{\nu N})^{\nu}}{J_{\nu+1}(j_{\nu N})} \cdot \frac{A^2}{(2\pi)^{\nu+1}} \\ &= \frac{1}{\phi_N\left(\frac{j_{\nu N}}{R}\right)} \frac{A^2}{(2\pi)^{\nu+1}}. \end{aligned} \tag{D.11}$$

Hence, we have the relation between $f_N(0)$ and $\phi_N(j_{\nu N}/R)$, that is,

$$f_N(0)\phi_N\left(\frac{j_{\nu N}}{R}\right) = \frac{A^2}{(2\pi)^{\nu+1}}, \tag{D.12}$$

where $\sigma_N = j_{\nu N}/R$ are the square root of the centered second moment of $\phi_N(\rho)$ (cf. Eq. (81)).

In this way, it follows that

$$\phi_N(\rho) = \frac{(-1)^{N-1}A j_{\nu N}\sqrt{2n}}{\Omega_n} \frac{J_\nu(R\rho)}{(R\rho)^\nu[(j_{\nu N})^2 - (R\rho)^2]} \tag{D.13}$$

$$= \begin{cases} \dfrac{(-1)^{N-1}A\sqrt{2n}}{j_{\nu N}(2\pi)^{\nu+1}} & : \rho = 0 \\ (-1)^{N-1}A \sqrt{\dfrac{n}{2}} \dfrac{J_{\nu+1}(j_{\nu N})}{\Omega_n (j_{\nu N})^\nu} = \dfrac{A^2}{(2\pi)^{\nu+1}} \cdot \dfrac{1}{f_N(0)} & : \rho = \dfrac{j_{\nu N}}{R} \\ 0 & : \rho = \dfrac{j_{\nu k}}{R} \begin{pmatrix} k = 1, 2, 3, \cdots \\ k \neq N \end{pmatrix} \\ \dfrac{(-1)^{N-1}A j_{\nu N}\sqrt{2n}}{\Omega_n} \dfrac{J_\nu(R\rho)}{(R\rho)^\nu[(j_{\nu N})^2 - (R\rho)^2]} & \\ & : \rho \neq \dfrac{j_{\nu k}}{R}\ (k = 0, 1, 2 \cdots) \end{cases} \tag{D.14}$$

$$\frac{(-1^N 2\sqrt{n}\, A j_{\nu N}}{\sqrt{\pi}\,\Omega_n} \frac{\cos(R\rho - (n-)1\pi/4)}{(R\rho)^{\nu+5/2}} : \rho \to \infty \tag{D.15}$$

APPENDIX E: CALCULATION FOR $s_1\sigma_1$

In order to obtain $s_1\sigma_1$, we first observe the Schafheitlin's reduction formula

$$
\begin{aligned}
(\mu + 2) \int^{Z} z^{\mu+2}\{J_\nu(z)\}^2\, dz
&= (\mu + 1)\left(\nu^2 - \frac{(\mu+1)^2}{4}\right)\int^{Z} z^{\mu}\{J_\nu(z)\}^2\, dz \\
&+ \frac{z^{\mu+1}}{2}\left[\left(zJ_\nu'(z) - \frac{\mu+1}{2}J_\nu(z)\right)^2\right. \\
&+ \left.\left(z^2 - \nu^2 + \frac{(\mu+1)^2}{4}\right)\{J_\nu(z)\}^2\right] \qquad \text{(E.1)}
\end{aligned}
$$

(Watson, 1952, p. 138). Substituting the recurrence formula (D.5) into Eq. (E.1), putting $\mu = 1$, and replacing z by at, then we have

$$
\begin{aligned}
\int^{t} t^3\{J_\nu(at)\}^2\, dt
&= \frac{t^2}{6a^2}\Big[\{a^2t^2 + 2(\nu^2 - \nu)\}\{J_\nu(at)\}^2 \\
&+ \{a^2t^2 + 2(\nu^2 - 1)\}\{J_{\nu+1}(at)\}^2 \qquad \text{(E.2)} \\
&- \frac{2(\nu - 1)}{at}\{a^2t^2 + 2(\nu^2 + \nu)\}\, J_\nu(at)J_{\nu+1}(at)\Big].
\end{aligned}
$$

From Eqs. (84), (E.2), and (81), we have

$$
\begin{aligned}
s_1^2 &\equiv \frac{n}{A^2R^n}\int_0^R r^{n+1}\{f_1(r)\}^2\, dr \\
&= \frac{2R^2}{\{J_{\nu+1}(j_\nu)\}^2}\int_0^1 r^3\{J_\nu(j_\nu r)\}^2\, dr \\
&= \left(\frac{R}{j_\nu}\right)^2 \cdot \frac{(j_\nu)^2 + 2(\nu^2 - 1)}{3} \\
&= \frac{(j_\nu)^2 + 2(\nu^2 - 1)}{3\sigma_1^2}. \qquad \text{(E.3)}
\end{aligned}
$$

Hence, it follows that

$$
s_1\sigma_1 = \sqrt{\frac{(j_\nu)^2 + 2(\nu^2 - 1)}{3}}. \qquad \text{(E.4)}
$$

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